ATKINSON SCIENCE

THEORY GUIDE

Admittance Method

4 Convection and Radiation

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23 October 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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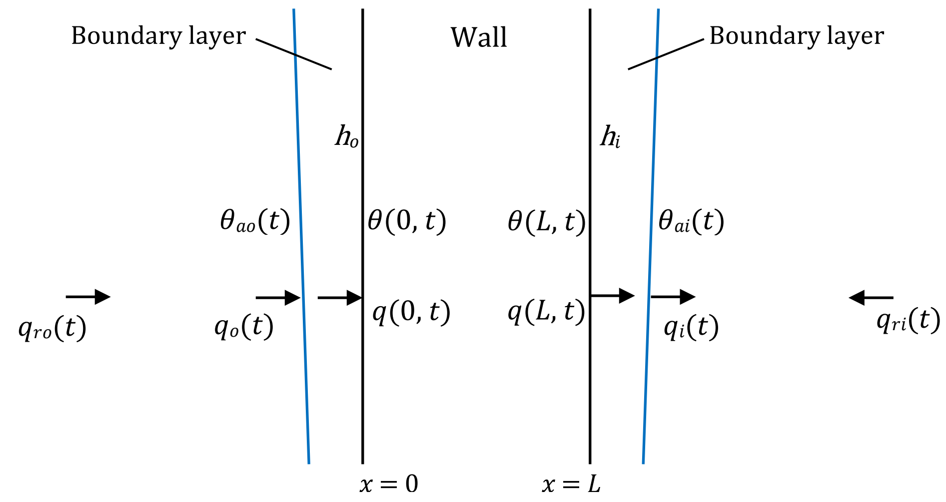
Figures

1 Convection and radiation at the faces of a wall

In the preceding report in this series, Ref. [1], we determined the heat fluxes at the inside and outside surfaces of a composite wall when a sinusoidal temperature variation is applied to one surface and the temperature on the other surface is held constant. For applications to buildings, we must also consider the convective and radiant heat exchange at the surfaces.

The convection heat transfer coefficient on the outside surface of the wall, $x = 0$, is denoted by h_o and the coefficient on the inside surface, $x = L$, is denoted by h_i, as shown in Figure 1. The time-varying air temperatures on the two sides are denoted by $\theta_{ao}(t)$ and $\theta_{ai}(t)$ and the time-varying net radiant heat gains on the two sides are denoted by $q_{ro}(t)$ and $q_{ri}(t)$.

Figure 1 Slab with boundary layers



1.1 Outside surface

The time-varying heat flux $q(0,t)$ [W m^{-2}] into the outside surface is

$$q(0,t) = h_o[\theta_{ao}(t) - \theta(0,t)] + q_{ro}(t) \quad (1.1)$$

We can regard the boundary layer on the surface as a slab. Since the slab has negligible thermal capacity, the heat flux into the slab must be the same as the heat flux leaving the slab, so

$$q_o(t) = q(0,t) \quad (1.2)$$

Rearranging (1.1) so that the wall surface temperature is on the left-hand side gives

$$\theta(0,t) = \theta_{ao}(t) + \frac{1}{h_o} q_{ro}(t) - \frac{1}{h_o} q(0,t) \quad (1.3)$$

In (1.3) the outside air temperature and the net absorbed radiant heat term can be combined and denoted by the *sol-air temperature* $\theta_{eo}(t)$. Thus:

$$\theta(0,t) = \theta_{eo}(t) - \frac{1}{h_o} q(0,t) \quad (1.4)$$

The Laplace transform of the heat flux equation (1.2) is

$$Q_o(s) = Q(0,s) \qquad (1.5)$$

and the Laplace transform of the temperature equation (1.4) is

$$\Theta(0,s) = \Theta_{eo}(s) - \frac{1}{h_o}Q(0,s) \qquad (1.6)$$

We can write (1.5) and (1.6) in matrix form, as follows:

$$\begin{bmatrix} \Theta_{eo}(s) \\ Q_o(s) \end{bmatrix} = \begin{bmatrix} 1 & 1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Theta(0,s) \\ Q(0,s) \end{bmatrix} \qquad (1.7)$$

We now have an equation for the heat flux in terms of the Laplace domain variable s. We shall now assume that the temperature excitation $\Theta_{eo}(s)$ is sinusoidal with angular frequency ω[rad s^{-1}]. Equation (1.7) is just as valid if we replace s with $j\omega$ where $j = \sqrt{-1}$. Equation (1.7) becomes

$$\begin{bmatrix} \Theta_{eo}(j\omega) \\ Q_o(j\omega) \end{bmatrix} = \begin{bmatrix} 1 & 1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Theta(0,j\omega) \\ Q(0,j\omega) \end{bmatrix} \qquad (1.8)$$

The sol-air temperature is sinusoidal, so

$$\Theta_{eo}(j\omega) = A_{eo}\sin(\omega t) = \mathrm{Im}\left(A_{eo}e^{j\omega t}\right) \qquad (1.9)$$

where A_{eo} is the amplitude of the sol-air temperature and Im means “the imaginary part of”.

If we replace $\Theta_{eo}(j\omega)$ in (1.8) with A_{eo} then (1.8) becomes

$$\begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} = \begin{bmatrix} 1 & 1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} \qquad (1.10)$$

where A_0 , Q_o and $Q(0)$ are complex constants. In (1.10) A_{eo} is used as a reference temperature and the phases of all the other quantities are determined with respect to the sol-air temperature $A_{eo}\sin(\omega t)$. If we put $A_0 = 0$ in (1.10) then

$$Q_o = Q(0) = h_o A_{eo}$$

so

$$q_o(t) = Q_o\sin(\omega t) = h_o A_{eo}\sin(\omega t) = h_o\mathrm{Im}\left(A_{eo}e^{j\omega t}\right) \qquad (1.11)$$

and

$$q(0,t) = Q(0)\sin(\omega t) = h_o A_{eo}\sin(\omega t) = h_o\mathrm{Im}\left(A_{eo}e^{j\omega t}\right) \qquad (1.12)$$

1.2 Inside surface

From Figure 1, the time-varying heat flux $q(L,t)$ out of the inside surface is

$$q(L,t) = h_i[\theta(L,t) - \theta_{ai}(t)] - q_{ri}(t) \quad (1.13)$$

Again, we can regard the boundary layer on the surface as a slab. Since the slab has negligible thermal capacity, the heat flux into the slab must be the same as the heat flux leaving the slab, so

$$q_i(t) = q(L,t) \quad (1.14)$$

Rearranging (1.13) so that the wall surface temperature is on the left-hand side gives

$$\theta(L,t) = \theta_{ai}(t) + \frac{1}{h_i} q_{ri}(t) + \frac{1}{h_i} q(L,t) \quad (1.15)$$

In (1.15) the indoor temperature and the net absorbed radiant heat term can be combined and denoted by the *environmental temperature* $\theta_{ei}(t)$. Thus:

$$\theta(L,t) = \theta_{ei}(t) + \frac{1}{h_i} q(L,t) \quad (1.16)$$

The Laplace transform of the heat flux equation (1.14) is

$$Q_i(s) = Q(L,s) \quad (1.17)$$

and the Laplace transform of the temperature equation (1.16) is

$$\Theta(L,s) = \Theta_{ei}(s) + \frac{1}{h_i} Q(L,s) \quad (1.18)$$

We can write (1.17) and (1.18) in matrix form, as follows:

$$\begin{bmatrix} \Theta(L,s) \\ Q(L,s) \end{bmatrix} = \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Theta_{ei}(s) \\ Q_i(s) \end{bmatrix} \quad (1.19)$$

We now have an equation for the heat flux in terms of the Laplace domain variable s. We shall now assume that the temperature excitation $\Theta_{eo}(s)$ is sinusoidal with angular frequency ω[rad s^{-1}]. Equation (1.19) is just as valid if we replace s with $j\omega$ where $j = \sqrt{-1}$. Equation (1.18) becomes

$$\begin{bmatrix} \Theta(L,j\omega) \\ Q(L,j\omega) \end{bmatrix} = \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Theta_{ei}(j\omega) \\ Q_i(j\omega) \end{bmatrix} \quad (1.20)$$

The temperature on the inside surface is sinusoidal, so

$$\Theta(L,j\omega) = A_L \sin(\omega t) = \mathrm{Im}\left(A_L e^{j\omega t}\right) \quad (1.21)$$

where A_L is the amplitude of the inside surface temperature and Im means "the imaginary part of".

If we replace $\Theta(L,j\omega)$ in (1.20) with A_L then (1.20) becomes

$$\begin{bmatrix} A_L \\ Q(L) \end{bmatrix} = \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} \quad (1.22)$$

where A_{ei} , Q_i and $Q(L)$ are complex constants. In (1.22) A_L is used as a reference temperature and the phases of all the other quantities are determined with respect to the inside surface temperature $A_L \sin(\omega t)$. If we put $A_{ei} = 0$ in (1.22) then

$$Q_i = Q(L) = h_i A_L$$

so

$$q_i(t) = Q_i \sin(\omega t) = h_i A_L \sin(\omega t) = h_i \mathrm{Im}\left(A_L e^{j\omega t}\right) \quad (1.23)$$

and

$$q(L,t) = Q(L) \sin(\omega t) = h_i A_L \sin(\omega t) = h_i \mathrm{Im}\left(A_L e^{j\omega t}\right) \quad (1.24)$$

1.3 Complex transmission matrix for a composite wall

A composite wall consists of n slabs of different materials in parallel. In the previous report in this series, Ref. [1], we showed that for a composite wall,

$$\begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} = \begin{bmatrix} \cosh M_1 & \dfrac{\sinh M_1}{N_1} \\ N_1 \sinh M_1 & \cosh M_1 \end{bmatrix} \begin{bmatrix} \cosh M_2 & \dfrac{\sinh M_2}{N_2} \\ N_2 \sinh M_2 & \cosh M_2 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cosh M_{n-1} & \dfrac{\sinh M_{n-1}}{N_{n-1}} \\ N_{n-1} \sinh M_{n-1} & \cosh M_{n-1} \end{bmatrix} \begin{bmatrix} \cosh M_n & \dfrac{\sinh M_n}{N_n} \\ N_n \sinh M_n & \cosh M_n \end{bmatrix} \begin{bmatrix} A_L \\ Q(L) \end{bmatrix} \quad (1.25)$$

The thickness L of the composite slab is

$$L = \sum_{i=1}^{n} l_i$$

where l_i is the thickness of the ith slab. We must calculate $M_1, M_2, \ldots, M_{n-1}, M_n$ and $N_1, N_2, \ldots, N_{n-1}, N_n$ before we can calculate the elements in the matrices and carry out the matrix multiplication.

We can regard the boundary layers as additional slabs. The heat output from one slab becomes the input to the next slab, so we can add the extra slabs to the beginning and the end of the sequence in (1.25):

$$\begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} = \begin{bmatrix} 1 & 1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh M_1 & \dfrac{\sinh M_1}{N_1} \\ N_1 \sinh M_1 & \cosh M_1 \end{bmatrix} \begin{bmatrix} \cosh M_2 & \dfrac{\sinh M_2}{N_2} \\ N_2 \sinh M_2 & \cosh M_2 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cosh M_{n-1} & \dfrac{\sinh M_{n-1}}{N_{n-1}} \\ N_{n-1} \sinh M_{n-1} & \cosh M_{n-1} \end{bmatrix} \begin{bmatrix} \cosh M_n & \dfrac{\sinh M_n}{N_n} \\ N_n \sinh M_n & \cosh M_n \end{bmatrix} \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} \quad (1.26)$$

Once we have carried out the matrix multiplications, we can write (1.26) in the compact form

$$\begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} \quad (1.27)$$

The square matrix in (1.27) is the complex transmission matrix for a composite wall with boundary layers on the outside and inside surfaces.

To apply (1.27) to the heating of buildings, we assume that the sol-air temperature variation is

$$\theta_{eo}(t) = A_{eo}\sin(\omega t) = \mathrm{Im}\left(A_{eo}e^{j\omega t}\right) \quad (1.28)$$

where A_{eo} is the amplitude of the sol-air temperature and Im means "the imaginary part of". The amplitude of the environmental temperature A_{eo} is set to zero. Eq. (1.27) reduces to

$$\begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} 0 \\ Q_i \end{bmatrix} \quad (1.29)$$

From (1.29) we obtain:

$$A_{eo} = z_2 Q_i$$

and

$$Q_o = z_4 Q_i$$

From these two equations we obtain

$$Q_i = \frac{A_{eo}}{z_2} \quad (1.30)$$

and

$$Q_o = A_{eo}\frac{z_4}{z_2} \quad (1.31)$$

The instantaneous heat flux on the indoor side is

$$q_i(t) = \mathrm{Im}\left[Q_i e^{j\omega t}\right] = \mathrm{Im}\left[\frac{A_{eo}}{z_2}e^{j\omega t}\right] \quad (1.32)$$

and the instantaneous heat flux on the outdoor side is

$$q_o(t) = \mathrm{Im}\left[Q_o e^{j\omega t}\right] = \mathrm{Im}\left[A_{eo}\frac{z_4}{z_2}e^{j\omega t}\right] \quad (1.33)$$

2 Example 1

The composite wall in Ref. [1] now has boundary layers on the inside and outside. On the outside surface the convection heat transfer coefficient is 25 W m^{-2} K^{-1}. On the inside surface the coefficient is 7.7 W m^{-2} K^{-1}. The sol-air temperature is sinusoidal with a mean of $0°C$ and an amplitude of $10°C$. The peak in sol-air temperature occurs at $3:00$ pm. The indoor environmental temperature is maintained at $0°C$. Calculate:

a. the steady thermal transmittance of the wall and boundary layers,
b. the transmission matrix for the wall and boundary layers,
c. the heat flux at the edge of the inside boundary, and
d. the heat flux at the edge of the outside boundary layer.

(a) The steady thermal transmittance (U value) of a composite wall with n layers and a boundary layer on each side is given by

$$\frac{1}{U} = \frac{1}{h_o} + \sum_{i=1}^{n} \frac{l_i}{k_i} + \frac{1}{h_i}$$

where l_i and k_i are the thickness and the thermal conductivity of the i^{th} layer, respectively. Hence

$$\frac{1}{U} = \frac{1}{25} + \frac{0.22}{0.77} + \frac{0.05}{0.042} + \frac{0.0125}{0.21} + \frac{1}{7.7} = 1.70558 \text{ m}^2 \text{ K W}^{-1}$$

so U is 0.58631 W m^{-2} K^{-1}.

(b) From Ref. [1], p. 12, the transmission matrix for the composite wall is

$$\boldsymbol{z} = \begin{bmatrix} (-4.43756 + j2.08549) & (-1.95249 + j4.42465) \\ (-47.0447 - j15.6345) & (-45.3168 + j18.7316) \end{bmatrix} \quad (2.1)$$

The transmission matrix for the outside boundary layer is

$$\begin{bmatrix} 1 & \frac{1}{h_o} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{25} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.04 \\ 0 & 1 \end{bmatrix} \quad (2.2)$$

The transmission matrix for the inside boundary layer is

$$\begin{bmatrix} 1 & \frac{1}{h_i} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{7.7} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.12987013 \\ 0 & 1 \end{bmatrix} \quad (2.3)$$

From (1.26), the transmission matrix for the wall, including the boundary layers, is

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} 1 & 0.04 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (-4.43756 + j2.08549) & (-1.95249 + j4.42465) \\ (-47.0447 - j15.6345) & (-45.3168 + j18.7316) \end{bmatrix} \begin{bmatrix} 1 & 0.12987 \\ 0 & 1 \end{bmatrix} \quad (2.4)$$

Multiplying the first two matrices on the right-hand side of (2.4) gives

$$\begin{bmatrix} 1 & 0.04 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (-4.43756 + j2.08549) & (-1.95249 + j4.42465) \\ (-47.0447 - j15.6345) & (-45.3168 + j18.7316) \end{bmatrix}$$

$$= \begin{bmatrix} (-6.31935 + j1.46011) & (-3.76516 + j5.17391) \\ (-47.0447 - j15.6345) & (-45.3168 + j18.7316) \end{bmatrix}$$

Multiplying this product by the third matrix on the right-hand side of (2.4) gives

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} (-6.31935 + j1.46011) & (-3.76516 + j5.17391) \\ (-47.0447 - j15.6345) & (-45.3168 + j18.7316) \end{bmatrix} \begin{bmatrix} 1 & 0.12987013 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-6.31935 + j1.46011) & (-4.58586 + j5.36354) \\ (-47.0447 - j15.6345) & (-51.4265 + j16.7011) \end{bmatrix} \quad (2.5)$$

This is the transmission matrix for the composite wall with boundary layers.

(c) The sinusoidal variation in sol-air temperature on the outdoor side is given by (1.28):

$$\theta_{eo}(t) = A_{eo}\sin(\omega t + \phi) = \mathrm{Im}\left(A_{eo}e^{j\omega t+\phi}\right)$$

For diurnal temperature variations, the angular speed ω is $2\pi \div (60 \times 60 \times 24) = 2\pi/86400$ rad s^{-1}. The peak in sol-air temperature occurs at $15{:}00$, so the offset ϕ must be $-2\pi(15-6)/24 = -0.75\pi$ rad. The amplitude A_{eo} of the temperature variation is 10°C, so (1.28) becomes

$$\theta_{eo}(t) = 10\sin(\omega t - 0.75\pi) = \mathrm{Im}\left[10e^{j(\omega t-0.75\pi)}\right] \quad (2.6)$$

Substituting the values of A_{eo} and ϕ into (1.32) gives the variation in heat flux on the indoor side:

$$q_i(t) = \mathrm{Im}\left[A_{eo}\frac{1}{z_2}e^{j(\omega t+\phi)}\right] = \mathrm{Im}\left[10\frac{1}{z_2}e^{j(\omega t-0.75\pi)}\right] \quad (2.7)$$

and substituting the values of A_{eo} and ϕ into (1.33) gives the variation in heat flux at the outdoor side:

$$q_o(t) = \mathrm{Im}\left[A_{eo}\frac{z_4}{z_2}e^{j(\omega t+\phi)}\right] = \mathrm{Im}\left[10\frac{z_4}{z_2}e^{j(\omega t-0.75\pi)}\right] \quad (2.8)$$

The complex number $1/z_2$ required in (2.7) is

$$\frac{1}{z_2} = \frac{1}{(-4.58586 + j5.36354)}$$

$$= \frac{(-4.58586 - j5.36354)}{(-4.58586 + j5.36354)(-4.58586 - j5.36354)}$$

$$= \frac{-4.58586 - j5.36354}{4.58586^2 - j^2 5.36354^2}$$

$$= \frac{-4.58586 - j5.36354}{49.7977}$$

$$= -0.09209 - j0.10771$$

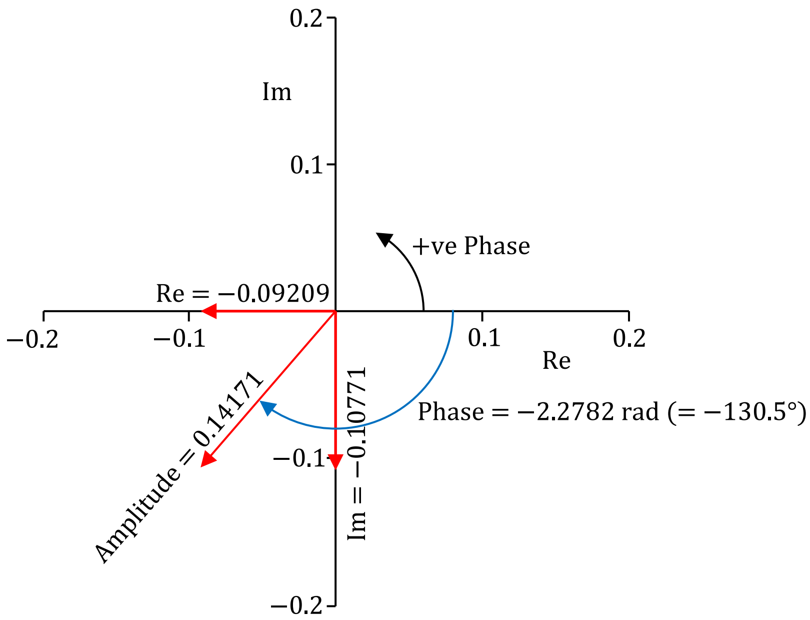
The complex number $1/z_2$ can be represented in the complex plane as shown in Figure 2. The amplitude of $1/z_2$ is

$$\text{Amplitude} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{(-0.09209)^2 + (-0.10771)^2} = 0.14171$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation on the inside surface of the wall will lag the temperature variation on the outside surface, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase in the clockwise (negative) direction from the positive Real axis gives

$$\text{Phase} = -2.2782 \text{ rad } (= -130.5°)$$

Figure 2 Amplitude and phase of $1/z_2$



We can now write $1/z_2$ as

$$\frac{1}{z_2} = 0.14171[\cos(-2.2782) + j\sin(-2.2782)]$$

$$= 0.14171e^{-j2.2782} \qquad (2.9)$$

Substituting (2.9) into (2.7) gives the heat flux at the indoor side:

$$q_i(t) = \mathrm{Im}\left[10 \times 0.14171e^{-j2.2782}e^{j(\omega t - 0.75\pi)}\right]$$

$$= \mathrm{Im}\left[1.4171e^{j(\omega t - 0.75\pi - 2.2782)}\right]$$

$$= 1.4171\sin(\omega t - 0.75\pi - 2.2782) \qquad (2.10)$$

The peak heat flux on the indoor side lags the peak sol-air temperature by 2.2782 rad (= 130.5°). In terms of hours, the lag is 24 hr × 130.5°/360° = 8.7 hr (8 hr 42 min).

(d) The complex number z_4/z_2 required in (2.8) is

$$\frac{z_4}{z_2} = (-51.4265 + j16.7011)(-0.09209 - j0.10771)$$

$$= 4.7359 + j5.5391 - j1.5380 - j^2 1.7989$$

$$= 6.5347 + j4.0011$$

The complex number z_4/z_2 can be represented in the complex plane as shown in Figure 3. The amplitude of z_4/z_2 is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(6.5347)^2 + (4.0011)^2} = 7.6623$$

We know the heat flux variation on the outdoor side will lead the variation in sol-air temperature, so the phase of the heat flux variation will be positive relative to the temperature variation. Measuring the phase in the anticlockwise (positive) direction from the positive Real axis gives

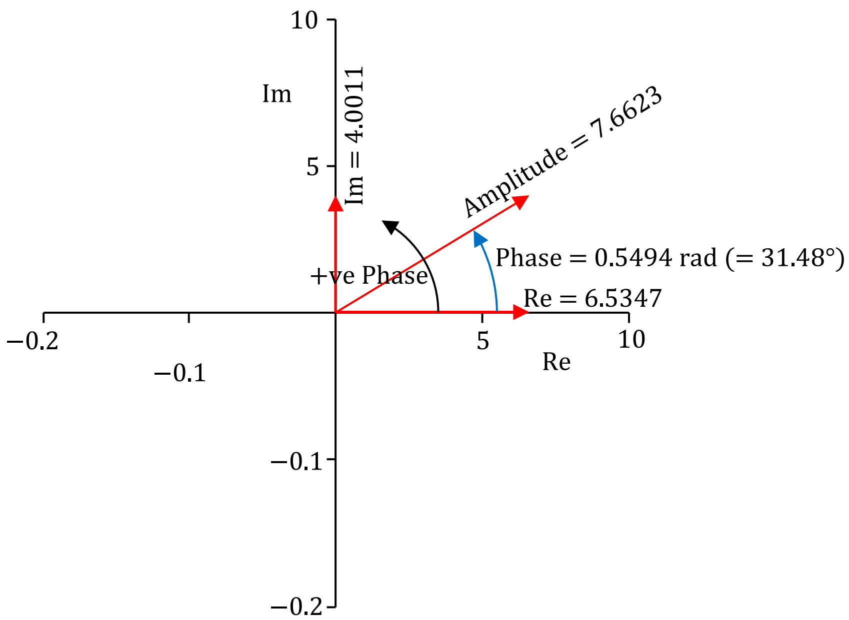
$$\text{Phase} = 0.5494 \text{ rad } (= 31.48°)$$

We can now write z_4/z_2 as

$$\frac{z_4}{z_2} = 7.6623(\cos 0.5494 + j \sin 0.5494)$$

$$= 7.6623 e^{j0.5494} \qquad (2.11)$$

Figure 3 Amplitude and phase of z_4/z_2



Substituting (2.11) into (2.8) gives

$$q_o(t) = \mathrm{Im}\left[10 \times 7.6623 e^{j0.5494} e^{j(\omega t - 0.75\pi)}\right]$$

$$= \mathrm{Im}\left[76.623 e^{j(\omega t - 0.75\pi + 0.5494)}\right]$$

$$= 76.623 \sin(\omega t - 0.75\pi + 0.5494) \qquad (2.12)$$

The peak heat flux on the outdoor side leads the peak sol-air temperature by 0.5494 rad $(= 31.48°)$. In terms of hours, the lead is 24 hr $\times\ 31.48°/360° = 2.099$ hr (2 hr 6 min).

3 Inverse transmission matrix

3.1 Homogeneous slab

For a homogeneous slab without boundary layers, the transmission matrix is

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix}$$

where

$$M = \sqrt{j\omega/\alpha}\, L$$

and

$$N = k\sqrt{j\omega/\alpha}$$

The determinant of the transmission matrix is $\cosh^2 M - \sinh^2 M = 1$, so the inverse transmission matrix is given by

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} \cosh(M) & -\dfrac{\sinh(M)}{N} \\ -N \sinh(M) & \cosh(M) \end{bmatrix} \quad (3.1)$$

3.2 Composite wall

For a composite wall made up of n slabs, we can use matrix multiplication to obtain the inverse transmission matrix:

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} \cosh(M_n) & -\dfrac{\sinh(M_n)}{N_n} \\ -N_n \sinh(M_n) & \cosh(M_n) \end{bmatrix} \begin{bmatrix} \cosh(M_{n-1}) & -\dfrac{\sinh(M_{n-1})}{N_n} \\ -N_n \sinh(M_n) & \cosh(M_n) \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cosh(M_2) & -\dfrac{\sinh(M_2)}{N_2} \\ -N_2 \sinh(M_2) & \cosh(M_2) \end{bmatrix} \begin{bmatrix} \cosh(M_1) & -\dfrac{\sinh(M_1)}{N_1} \\ -N_1 \sinh(M_1) & \cosh(M_1) \end{bmatrix} \quad (3.2)$$

Note that we have used the rule $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ in (3.2), where $\mathbf{A}$ and $\mathbf{B}$ are square matrices. We must calculate M_1, M_2, …, M_{n-1}, M_n and N_1, N_2, …, N_{n-1}, N_n before we can calculate the elements in the matrices and carry out the matrix multiplication.

3.3 Convection and radiation at the surfaces

The inverse of Eq. (1.10) is

$$\begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} = \begin{bmatrix} 1 & -1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} \quad (3.3)$$

and the inverse of Eq. (1.22) is

$$\begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} = \begin{bmatrix} 1 & -1/h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_L \\ Q(L) \end{bmatrix} \quad (3.4)$$

To account for convection and radiation, we add (3.3) and (3.4) to the end and to the beginning of the sequence in (3.2):

$$\begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} = \begin{bmatrix} 1 & -1/h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh(M_n) & -\dfrac{\sinh(M_n)}{N_n} \\ -N_n \sinh(M_n) & \cosh(M_n) \end{bmatrix} \begin{bmatrix} \cosh(M_{n-1}) & -\dfrac{\sinh(M_{n-1})}{N_n} \\ -N_n \sinh(M_n) & \cosh(M_n) \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cosh(M_2) & -\dfrac{\sinh(M_2)}{N_2} \\ -N_2 \sinh(M_2) & \cosh(M_2) \end{bmatrix} \begin{bmatrix} \cosh(M_1) & -\dfrac{\sinh(M_1)}{N_1} \\ -N_1 \sinh(M_1) & \cosh(M_1) \end{bmatrix} \begin{bmatrix} 1 & -1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} \quad (3.5)$$

After multiplying the matrices, we can write (3.5) as:

$$\begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} \quad (3.6)$$

To apply (3.6) to the heating of buildings, we assume that the inside environmental temperature is

$$\theta_{ei}(t) = A_{ei} \sin(\omega t) = \mathrm{Im}\left(A_{ei} e^{j\omega t}\right) \quad (3.7)$$

where A_{ei} is the amplitude of the environmental temperature and Im means "the imaginary part of". If the sol-air temperature is maintained at zero so $A_{eo} = 0$, then (3.6) reduces to

$$\begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} 0 \\ Q_o \end{bmatrix} \quad (3.8)$$

where Q_i and Q_o are complex constants. In (3.8), A_{ei} is used as a reference temperature and the phases of the other quantities are determined with respect to the environmental temperature $\theta_{ei}(t)$ given by (3.7).

On the outdoor side the variation in heat flux is

$$q_o(t) = \mathrm{Im}\left[Q_o e^{j\omega t}\right] = \mathrm{Im}\left[A_{ei} \frac{1}{Z_2} e^{j\omega t}\right] \quad (3.9)$$

and on the indoor side the variation in heat flux is

$$q_i(t) = \mathrm{Im}\left[Q_i e^{j\omega t}\right] = \mathrm{Im}\left[A_{ei} \frac{Z_4}{Z_2} e^{j\omega t}\right] \quad (3.10)$$

4 Example 2

For the composite wall in Example 1, the environmental temperature is sinusoidal with a mean of 0°C and an amplitude of 5°C. The peak in environmental temperature occurs at 12:00 noon. The sol-air temperature is constant at 0°C. Calculate the heat flux (a) at the edge of the outside boundary, and (b) at the edge of the inside boundary layer.

(a) From Example 1, the complex transmission matrix for the composite wall with boundary layers is

$$\boldsymbol{z} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} (-6.31935 + j1.46011) & (-4.58586 + j5.36354) \\ (-47.0447 - j15.6345) & (-51.4265 + j16.7011) \end{bmatrix}$$

The inverse of this matrix is

$$\boldsymbol{Z} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} (-51.4321 + j16.6913) & (4.58715 - j5.36282) \\ (47.0435 + j15.6448) & (-6.31991 + j1.45887) \end{bmatrix}$$

The sinusoidal temperature variation on the indoor side is given by (3.7):

$$\theta_{ei}(t) = A_{ei}\sin(\omega t + \varphi) = \mathrm{Im}\left(A_{ei}e^{j\omega t + \varphi}\right)$$

The peak in temperature occurs at 12:00, so the offset φ must be $-2\pi(12-6)/24 = -0.5\pi$ rad. The amplitude A_{ei} of the temperature variation is 5°C, so (3.7) becomes

$$\theta_{ei}(t) = 5\sin(\omega t - 0.5\pi) = \mathrm{Im}\left[5e^{j(\omega t - 0.5\pi)}\right] \qquad (4.1)$$

Substituting the values of A_{ei} and φ into (3.9) gives the variation in heat flux on the outdoor side:

$$q_o(t) = \mathrm{Im}\left[5\frac{1}{Z_2}e^{j(\omega t - 0.5\pi)}\right] \qquad (4.2)$$

and substituting the values of A_{ei} and φ into (3.10) gives the variation in heat flux on the indoor side:

$$q_i(t) = \mathrm{Im}\left[5\frac{Z_4}{Z_2}e^{j(\omega t - 0.5\pi)}\right] \qquad (4.3)$$

The term $1/Z_2$ required in (4.2) is

$$\frac{1}{Z_2} = \frac{1}{4.58715 - j5.36282}$$

$$= \frac{4.58715 + j5.36282}{(4.58715 - j5.36282)(4.58715 + j5.36282)}$$

$$= \frac{4.58715 + j5.36282}{4.58715^2 + 5.36282^2}$$

$$= 0.092108 + j0.10768$$

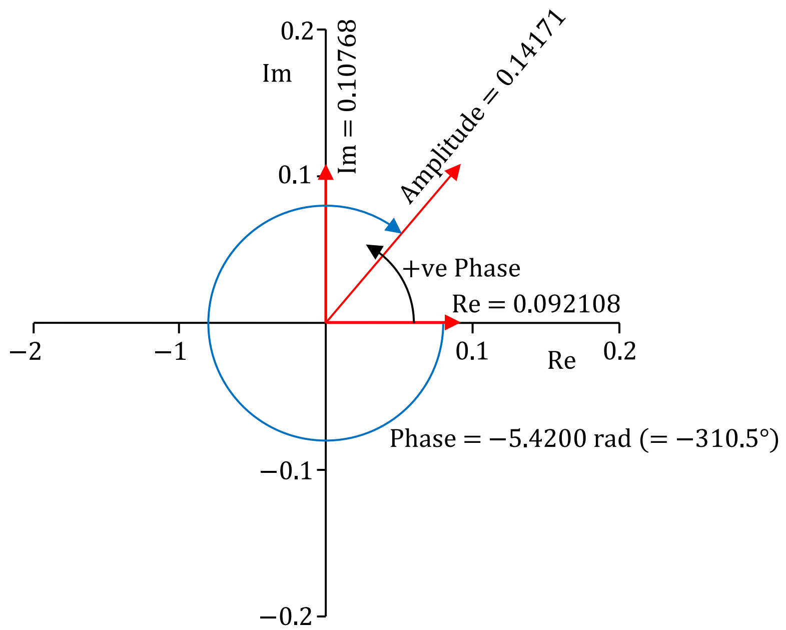
The complex number $1/Z_2$ can be represented in the complex plane as shown in Figure 4. The amplitude of $1/Z_2$ is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{0.092108^2 + 0.10768^2} = 0.14170$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation on the outside surface of the wall will lag the temperature variation on the inside surface, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase in the clockwise (negative) direction from the positive Real axis gives

$$\text{Phase} = \ -5.4200 \text{ rad } (= -310.5°)$$

Figure 4 Amplitude and phase of $1/Z_2$



We can now write $1/Z_2$ as

$$\frac{1}{Z_2} = 0.14171[\cos(-5.4200) + j\sin(-5.4200)]$$

$$= 0.14171e^{-j5.4200} \qquad (4.4)$$

Substituting (4.4) into (4.2) gives

$$q_o(t) = \text{Im}\left[5 \times 0.14171e^{-j5.4200}e^{j(\omega t - 0.5\pi)}\right]$$

$$= \text{Im}\left[0.7085e^{j(\omega t - 0.5\pi - 5.4200)}\right]$$

$$= 0.7085\sin(\omega t - 0.5\pi - 5.4200) \qquad (4.5)$$

The peak heat flux of $+0.7085$ W m^{-2} on the outdoor side lags the peak environmental temperature by 5.4200 rad $(= 310.5°)$. In terms of hours, the lag is 24 hr $\times$ $310.5°/360°$ = 20.7 hr (20 hr 42 min). This is the time difference between the positive peak in $\theta_{ei}(t)$ and the positive peak in $q_o(t)$. We would expect a positive peak in the environmental temperature to give rise to a *negative* peak in heat flux on the outdoor side, because heat flows in the negative x direction when $\theta_{ei}(t)$ is positive. The time lag between the negative peak in heat flux and the positive peak in temperature is 20 hr 42 min $- 12$ hr $=$ 8 hr 42 min, which is much shorter.

Notice that 8 hr 42 min is the same as the time between the sol-air temperature $\theta_{eo}(t)$ and the corresponding heat flux on the indoor side. The amplitude term 0.14171 is also the same.

(b) The term Z_4/Z_2 required in (4.3) is

$$\frac{Z_4}{Z_2} = (-6.31991 + j1.45887)(0.092108 + j0.10768)$$

$$= -0.5821143 - j0.6805279 + j0.1343736 + j^2 0.1570911$$

$$= -0.7392054 - j0.5461543$$

The complex number Z_4/Z_2 can be represented in the complex plane as shown in Figure 5. The amplitude of Z_4/Z_2 is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(-0.73921)^2 + (-0.54615)^2} = 0.9190806$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation on the inside surface of the wall will lead the temperature variation on the inside surface, so the phase of the heat flux variation will be positive relative to the temperature variation. Measuring the phase in the anticlockwise (positive) direction from the positive Real axis gives

$$\text{Phase} = \ 3.7779 \text{ rad } (= 216.46°)$$

We can now write Z_4/Z_2 as

$$\frac{Z_4}{Z_2} = 0.91908[\cos 3.7779 + j \sin 3.7779]$$

$$= 0.91908e^{j3.7779} \qquad (5.16)$$

Substituting (5.16) into (5.13) gives

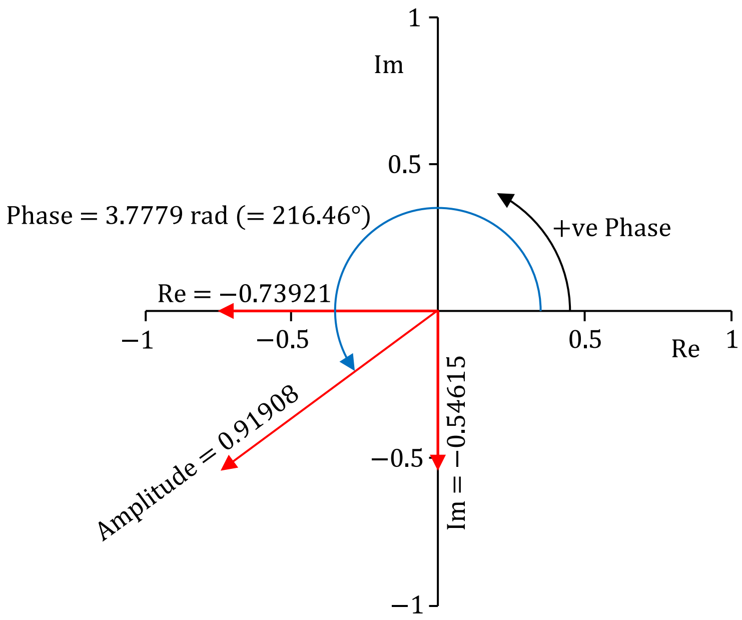
$$q_i(t) = \text{Im}\left[5 \times 0.91908e^{j3.7779}e^{j(\omega t - 0.5\pi)}\right]$$

$$= \text{Im}\left[4.5954e^{j(\omega t - 0.5\pi + 3.7779)}\right]$$

$$= 4.5954 \sin(\omega t - 0.5\pi + 3.7779) \qquad (5.17)$$

The peak heat flux of $+4.5954\ \mathrm{W\ m^{-2}}$ on the indoor side leads the peak environmental temperature by 3.7779 rad ($= 216.46°$). In terms of hours, the lead is 24 hr $\times$ $216.46°/360°$ $=$ 14.431 hr (14 hr 26 min). This is the time difference between the positive peak in $\theta_{ei}(t)$ and the positive peak in $q_i(t)$. We would expect a positive peak in $\theta_{ei}(t)$to be caused by a *negative* peak $q_i(t)$. The time lead between the negative peak in heat flux and the positive peak in environmental temperature is 14 hr 26 min $-$ 12 hr $=$ 2 hr 26 min, which is much shorter.

Figure 5 Amplitude and phase of Z_4/Z_2



5 References

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